Advanced Studies in Contemporary Mathematics
34 (2024), No. 4, pp. 267 - 271
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2024.34.4.267

A STUDY ON CAPSTONE UNITS WITH RIORDAN'S FACTORIAL POWERS

[1]YEŞIM AKÜZÜM, [2]ANTHONY G. SHANNON AND [3]ÖMÜR DEVECI

Abstract. This is partly a discursive paper that raises issues related to undergraduate capstone subjects and partly an expository paper that describes some examples. The context is notation as a tool of thought, particularly in relation to falling and rising factorial powers in the framework of the finite difference calculus.

1. Introduction

Capstone subjects are units at the end of a degree that builds on the previous subjects within the degree on a topic that they could be expected to undertake in their first year in the workforce after their degree. The topics in the project often come from experienced practitioners in the relevant industry and are frequently carried out in small groups, akin to the manner in which their field of study operates in the world of work. They also act as a gentle introduction to the elements of research and development in that particular field.

The purpose of this study, as in a previous work [3], is to provide ideas for undergraduate capstone units, where shrewd guessing can provide a gentle introduction to mathematical research [10]. In particular, devotees of number theory and discrete mathematics need no introduction to John Riordan (1903-1988) especially in references to his long-term collaboration with Leonard Carlitz (1907-1999) [2] where their work lends itself to inspiring undergraduates to try their hands at genuine mathematical research.

In the spirit of Riordan and Carlitz for notation as a tool of thought in mathematics (and music) [5], this study outlines some aspects of Riordan's work with falling and rising factorials. Much of this came to fruition during and after a conference at Chapel Hill in 1969 after which Donald Knuth's underline and overline notation came to be relatively

2000 *Mathematics Subject Classification.* 05A10, 11B39, 97A99.
Key words and phrases. Analogies, Appell sets, difference operators, Fibonacci numbers and polynomials, Lucas numbers, falling factorial coefficients, partitions, permutations, rising factorial coefficients.

popular [6]. Riordan had been involved with attempts for uniformity of notation for many years.

The falling factorial is defined as the polynomial

$$n^{\underline{r}} = n\,(n-1)\,(n-2)\ldots(n-r+1) = \prod_{k=0}^{n-1}(n-k) \tag{1.1}$$

for example,

$$u_{n+1}(x) = xu_n(x) + \sum_{j=0}^{n} n^{\underline{j}} v_{j+1} u_{n-j}(x).$$

in which $u_n(x)$ is a generalized Fibonacci polynomial defined [12] by

$$u_n(x) = \sum_{k=0}^{n} u_{n-k}\frac{n!}{k!}x^k.$$

The rising factorial is defined as the

$$n^{\bar{r}} = n\,(n+1)\,(n+2)\ldots(n+r-1) = \prod_{k=0}^{n-1}(n+k). \tag{1.2}$$

The coefficients in the expansions of both (1.1) and (1.2) are Stirling numbers of the first kind, and their inverses are Stirling numbers of the second kind as in Riordan [7]. An example of rising factorials occurs in the Saalschütz formula [11], with notation as a tool of thought.

2. **Falling Factorials**

The falling factorial $n^{\underline{r}}$ is an r *permutation* of n distinct objects, such that [7]

$$\nabla n^{\underline{r}} = n^{\underline{r}} - (n-1)^{\underline{r}} = r\,(n-1)^{\underline{r-1}}. \tag{2.1}$$

Furthermore, Riordan [9] proved that the falling factorial occupies a central position in the finite difference calculus because from (2.1) it follows that $\nabla x^{\underline{n}} = nx^{\underline{n-1}}$, an analogous result occurs in the next section and likewise with the other results in that section. Now it can be readily confirmed that the connection with the binomial coefficient is given by

$$\begin{aligned}\binom{n}{r} &= \frac{n!}{r!\,(n-r)!} \\ &= \frac{n\,(n-1)\ldots(n-r+1)\,(n-r)\ldots 3.2.1}{r\,(r-1)\ldots(r-r+1)\,(n-r+1)\,(n-r)\ldots 3.2.1} \\ &= = \frac{n\,(n-1)\ldots(n-r+1)}{r\,(r-1)\ldots(r-r+1)}\end{aligned}$$

that is,

$$\binom{n}{r} = \frac{n^{\underline{r}}}{r^{\underline{r}}}. \tag{2.2}$$

3. Rising Factorials

The rising factorial $n^{\overline{r}}$ is the number of partitions of an r-element set into n ordered sequences. By analogy with (2.1), we can show that

$$\nabla n^{\overline{r}} = n^{\overline{r}} - (n-1)^{\overline{r-1}} = rn^{\overline{r-1}}. \tag{3.1}$$

Proof.

$$\begin{aligned}(n-1)^{\overline{r-1}} + rn^{\overline{r-1}} &= (n-1)\,n\,(n+1)\ldots(n+r-2) + rn\,(n+1)\,(n+2)\ldots(n+r-2) \\ &= n\,(n+1)\ldots(n+r-2)\,(n+r-1) \\ &= n^{\overline{r}}.\end{aligned}$$

□

An equation of the form $n^{\overline{r}} - (n-1)^{\overline{r-1}} = rn^{\overline{r-1}}$ is an extension of the criteria for an Appell set [4]. Riordan [8] implied variations of alternative combinatorial coefficients that have Appell criteria; for example, consider a rising factorial analogue of (2.2):

$$C\,(n,j;r) = \frac{n^{\overline{r}}}{j^{\overline{r}}} \tag{3.2}$$

so that

$$C\,(-n,-r:r) = \binom{n}{r}$$

and the difference operator can be

$$\nabla C\,(n,r;r) = C\,(n,r;r) - C\,(n-1,r;r) = C\,(n,r+1;r-1)\,. \tag{3.3}$$

4. Concluding Comments

We can also define rising factorial analogues of other special functions; for instance, a rising factorial analogue of the exponential function could be

$$e\left(n,r;x\right)=\sum_{n=0}^{\infty}\frac{x^{n}}{\left(n+r\right)^{\overline{r}}}. \tag{4.1}$$

Then, for example

$$\begin{aligned} e\left(n,r;1\right) &= \prod_{s=1}^{r}\sum_{n=0}^{\infty}\left(s+r\right)^{-1} \\ &= \prod_{s=1}^{r}\zeta\left(1,s\right) \end{aligned} \tag{4.2}$$

where $\zeta\left(j,s\right)$ denotes a generalized zeta function. Riordan [9] shown how these factorial powers occupy a central position in the finite difference calculus. Carlitz [1] has produced similar results with q-series analogues of the binomial coefficients defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q=\frac{q_n}{q_r q_{n-r}} \tag{4.3}$$

in which q_n is defined formally as $q_n=\left(1-q\right)\left(1-q^2\right)\ldots\left(1-q^n\right)$.

References

[1] Carlitz, L., *Note on a q-identity.* Math. Scand., 3 (1955), no. 2, 281-282.

[2] Carlitz, L. and Riordan. J., *Two element lattice permutation numbers and their q-generalization.* Duke Math. J., 31 (1964), no. 1, 317-388.

[3] Deveci, O. and Shannon A. G., *A note on balanced incomplete block designs and projective geometry.* Int. J. Math. Educ. Sci. Technol., 52 (2020), no. 5, 807-813.

[4] Irene, F. M. and Malonek, H. R., *Generalized Exponentials through Appell sets in* R^{n+1} *and Bessel functions.* Amer. Inst. Phys. Conf. Proc., 936 (2007), 738-741.

[5] Iverson, K. E., *Notation as a Tool of Thought.* Comm. Assoc. Computing Machinery, 23 (1980), no. 8, 444-465.

[6] Knuth, D. E., *Two notes on notation.* Amer. Math. Monthly, 99 (1992), no. 5, 403-422.

[7] Riordan, J., *An Introduction to Combinatorial Analysis.* Wiley, New York, p.3, 1958.

[8] Riordan, J., *Inverse Relations and Combinatorial Identities.* Amer. Math. Monthly, 71 (1964), no. 5, 485-498.

[9] Riordan, J., *Combinatorial Identities.* Wiley, New York. pp. 45, 202, 1968.

[10] Shannon, A. G., *Shrewd guessing in problem-solving.* Int. J. Math. Educ. Sci. Technol., 22 (1991), no. 1, 144-147.

[11] Shannon, A. G., *Saalschütz's Theorem and Rising Binomial Coefficients - Type 2*. Notes Number Theory Discrete Math., 26 (2020), no. 2, 142-147.

[12] Shannon, A. G. and Deveci, O., *Some generalized Fibonacci and Hermite polynomials*. JP Journal Algebra, Number Theory Appl., 40 (2018), no. 4, 419-427.

[1,3]Department of Mathematics, Faculty of Science and Letters, Kafkas University 36100, Turkey, [2]Warrane College, the University of New South WalesKensington, NSW 2033, Australia

Email address: yesim_036@hotmail.com; t.shannon@warrane.unsw.edu.au; odeveci36@hotmail.com